ANTI-DERIVATIVES

Ref: Complex Variables by James Ward Brown and Ruel V. Churchil

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42. ANTI-DERIVATIVES

Formula

1) Continuity

2) $F'(z) = \lim_{\Delta z \to 0} \dfrac{F(z+\Delta z)-F(z)}{\Delta z}$

Definition

The anti-derivative of a continuous function f in a domain D, is a function F such that $F'(z) = f(z)\ \forall\ z$ in D.

Remarks

1) The anti-derivative is an analytic function.

2) The anti-derivative of a given function f is unique.

Theorem

Suppose that a function f(z) is continuous on a domain D. If any one of the following statements is true, then so are the others:

i) f(x) has an anti-derivative F(z) in D.

ii) The integrals of f(z) along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value. i.e., the integration is independent of the path in D.

iii) The integrals of f(z) around closed contours lying entirely in D all have values zero.

Proof

To prove (i) $\Rightarrow$ (ii)

Assume f(z) has an anti-derivative F(z) in D.

$\Rightarrow F'(z) = f(z)$

To Prove

The integrals of f(z) along contours in D all have the same value.

If a contour C from z_1 to z_2 is $z = z(t)$ $(a \le t \le b)$ then

$$\frac{d}{dt} F(z(t)) = F'[z(t)]z'(t)$$

$$= f[z(t)]z'(t)$$

Now, $\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

$$= \int_a^b \frac{d}{dt}\left(F\left[z(t)\right]\right) dt$$

$$\therefore \int_C f(z)\,dz = [F[z(t)]\ dt]_a^b$$

$= F[z(b)] – F[z(a)]$

$= F(z_2) – F(z_1)$

$\therefore$ The integrals of f(z) along contours in D all have the same value and this is also valid when C is any contour, not necessarily a smooth one, that lies in D.

For, if C consists of finite number of smooth arcs C_k (k = 1, 2, . . . , n), each C_k extending from z_k to z_{k+1} then $\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$

$$= \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)]$$

$$= F(z_{n+1}) - F(z_1)$$

$\therefore$ (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii)

Assume that the integrals of f(z) along contours in D all have the same value.

To prove

The integrals of f(z) around closed contours in D have value 0.

We let z_1 and z_2 denote any two points on a closed contour C lying in D and form paths with initial point z_1 and final point z_2 such that $C = C_1 - C_2$.

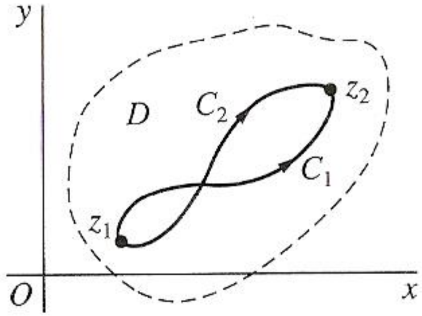

FIGURE 48

By assumption

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Now, $\displaystyle\int_{C} f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$\therefore$ The integral of f(z) along any closed contour C is 0.

Now to prove (iii) $\Rightarrow$ (ii)

i.e., To prove (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

Let the integral of f(z) along any closed contour C be 0.

Let C_1 and C_2 be any two contours lying in D from a point z_1 to z_1 and

we also have $\int_{C_1 - C_2} f(z)\,dz = 0.$

$$\Rightarrow \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Define $F(z) = \int_{z_0}^{z} f(s)\,ds \text{ on } D$

We have to show that $F'(z) = f(z)$.

Let $(z + \Delta z)$ be any point, distinct from z, lying in some neighborhood of z that is small enough to be contained in D.

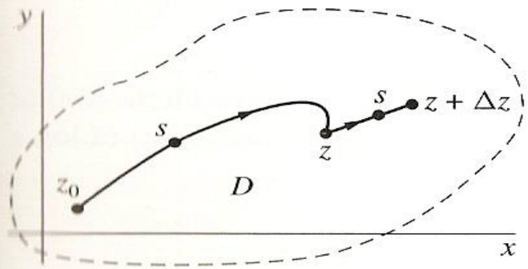

FIGURE 49

$$\text{Now, } F(z + \Delta z) - F(z) = \int_{z_0}^{z_0 + \Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

$$= \int_{z}^{z + \Delta z} f(s)\,ds$$

where the path of integration from z to z + Δz may be selected as a line segment.

Now, $$\int_{z}^{z+\Delta z} ds = \Delta z$$

$$\Rightarrow \int_{z}^{z+\Delta z} f(z)\, ds = f(z)\, \Delta z$$

$$\Rightarrow \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\, ds = f(z)$$

Now, $$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]\, ds$$

Given: f is continuous at z.

$\Rightarrow$ given $\in > 0$, there exists $\delta > 0$ such that $|f(s) - f(z)| < \varepsilon$ whenever $|s - z| < \delta$.

Now, $z + \Delta z$ is close to $z \Rightarrow |\Delta z| < \delta$.

$$\therefore \left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \frac{1}{\Delta z}\left|\int_z^{z+\Delta z}[f(s)-f(z)]\,ds\right| < \frac{1}{\Delta z}\left|\int_z^{z+\Delta z}\varepsilon\,ds\right|$$

$$= \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon$$

$$\therefore \lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

$\Rightarrow F'(z) = f(z)$

$\Rightarrow$ The anti-derivative of f(z) if F(z). ■